

FALTINGS' LOCAL-GLOBAL PRINCIPLE FOR THE FINITENESS OF LOCAL COHOMOLOGY MODULES OVER NOETHERIAN RINGS

ALI AKBAR MEHRVARZ*, REZA NAGHIPOUR AND MONIREH SEDGHI

ABSTRACT. Let R denote a commutative Noetherian (not necessarily local) ring, \mathfrak{a} an ideal of R and M a finitely generated R -module. The purpose of this paper is to show that $f_{\mathfrak{a}}^n(M) = \inf\{0 \leq i \in \mathbb{Z} \mid \dim H_{\mathfrak{a}}^i(M)/N \geq n \text{ for any finitely generated submodule } N \subseteq H_{\mathfrak{a}}^i(M)\}$, where n is a non-negative integer and the invariant $f_{\mathfrak{a}}^n(M) := \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \text{Supp } M/\mathfrak{a}M \text{ and } \dim R/\mathfrak{p} \geq n\}$ is the n -th finiteness dimension of M relative to \mathfrak{a} . As a consequence, it follows that the set

$$\text{Ass}_R(\oplus_{i=0}^{f_{\mathfrak{a}}^n(M)} H_{\mathfrak{a}}^i(M)) \cap \{\mathfrak{p} \in \text{Spec } R \mid \dim R/\mathfrak{p} \geq n\}$$

is finite. This generalizes the main result of Quy [10], Brodmann-Lashgari [5] and Asadollahi-Naghipour [1].

1. INTRODUCTION

Let R be a commutative Noetherian ring, \mathfrak{a} an ideal of R , and M a finitely generated R -module. An important theorem in local cohomology is Faltings' Local-global Principle for the finiteness of local cohomology modules ([8, Satz 1]) which states that for a positive integer r the $R_{\mathfrak{p}}$ -module $H_{\mathfrak{a}R_{\mathfrak{p}}}^i(M_{\mathfrak{p}})$ is finitely generated for all $i < r$ and for all $\mathfrak{p} \in \text{Spec } R$ if and only if the R -module $H_{\mathfrak{a}}^i(M)$ is finitely generated for all $i < r$.

Another formulation of Faltings' Local-global Principle, particularly relevant for this paper, is in terms of the finiteness dimension $f_{\mathfrak{a}}(M)$ of M relative to I , where

$$f_{\mathfrak{a}}(M) := \inf\{i \in \mathbb{N} : H_{\mathfrak{a}}^i(M) \text{ is not finitely generated}\},$$

with the usual convention that the infimum of the empty set of integers is interpreted as ∞ . Bahmanpour et al., in [4], introduced the notion of the n -th finiteness dimension $f_{\mathfrak{a}}^n(M)$ of M relative to I by

$$f_{\mathfrak{a}}^n(M) := \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) : \mathfrak{p} \in \text{Supp } M/\mathfrak{a}M \text{ and } \dim R/\mathfrak{p} \geq n\}.$$

More recently, Asadollahi and Naghipour in [1], introduced the class of *in dimension* $< n$ modules, and they showed that, if (R, \mathfrak{m}) is a complete local ring, \mathfrak{a} an ideal of R and M a finitely generated R -module, then for any $n \in \mathbb{N}_0$,

$$f_{\mathfrak{a}}^n(M) = \inf\{0 \leq i \in \mathbb{Z} \mid H_{\mathfrak{a}}^i(M) \text{ is not in dimension } < n\}.$$

In this paper, we eliminate the complete local hypothesis entirely by proving the following:

Key words and phrases. Associated primes, Faltings' local-global principle, Local cohomology.

2010 *Mathematics Subject Classification:* 13D45, 14B15, 13E05.

*Corresponding author: e-mail: amehrvarz2013@gmail.com (Ali Akbar Mehrvarz).

Proposition 1.1. *Let R be a Noetherian ring, \mathfrak{a} an ideal of R and M a finitely generated R -module. Then for any $n \in \mathbb{N}_0$,*

$$f_{\mathfrak{a}}^n(M) = \inf\{0 \leq i \in \mathbb{Z} \mid H_{\mathfrak{a}}^i(M) \text{ is not in dimension } < n\}.$$

The proof of Proposition 1.1 is given in Theorem 2.10. Pursuing this point of view further we establish the following consequence of Proposition 1.1.

Theorem 1.2. *Let R be a Noetherian ring and \mathfrak{a} an ideal of R . Let n be a non-negative integer and M a finitely generated R -module. Then the R -modules $\text{Ext}_R^j(R/\mathfrak{a}, H_{\mathfrak{a}}^i(M))$ are in dimension $< n$ for all $i < f_{\mathfrak{a}}^n(M)$ and all integers j . Moreover, the R -modules $\text{Ext}_R^j(N, H_{\mathfrak{a}}^{f_{\mathfrak{a}}^n(M)}(M))$ are in dimension $< n$, for each finitely generated R -module N with support in $V(\mathfrak{a})$ and $j = 0, 1$.*

As a consequence of Proposition 1.1 and Theorem 1.2, we derive the following, which is a generalization of the main result of Quy [10, Theorem 3.2] and Brodmann-Lashgari [5, Theorem 2.2].

Corollary 1.3. *Let R be a Noetherian ring, \mathfrak{a} an ideal of R and M a finitely generated R -module. Then for any $n \in \mathbb{N}_0$, the set*

$$\text{Ass}_R(\oplus_{i=0}^{f_{\mathfrak{a}}^n(M)} H_{\mathfrak{a}}^i(M)) \cap \{\mathfrak{p} \in \text{Spec } R \mid \dim R/\mathfrak{p} \geq n\},$$

is finite.

Another consequence of Proposition 1.1 shows that the first non-minimax (resp. non-skinny) local cohomology modules $H_{\mathfrak{a}}^i(M)$ of a finitely module M over a Noetherian ring R with respect to an ideal \mathfrak{a} is $f_{\mathfrak{a}}^n(M) = \inf\{0 \leq i \in \mathbb{Z} \mid H_{\mathfrak{a}}^i(M) \text{ is not in dimension } < 1\}$ (resp. $f_{\mathfrak{a}}^n(M) = \inf\{0 \leq i \in \mathbb{Z} \mid H_{\mathfrak{a}}^i(M) \text{ is not in dimension } < 2\}$).

Throughout this paper, R will always be a commutative Noetherian ring with non-zero identity and \mathfrak{a} will be an ideal of R . For an R -module L , the i -th local cohomology module of L with support in $V(\mathfrak{a})$ is defined as:

$$H_{\mathfrak{a}}^i(L) = \varinjlim_{n \geq 1} \text{Ext}_R^i(R/\mathfrak{a}^n, L).$$

Local cohomology was defined and studied by Grothendieck. We refer the reader to [6] or [9] for more details about local cohomology.

For a non-negative integer n , an R -module M is said to be *in dimension* $< n$ if there is a finitely generated submodule N of M such that $\dim M/N < n$. By a *skinny* or *weakly Laskerian* module, we mean an R -module M such that the set $\text{Ass}_R M/N$ is finite, for each submodule N of M (cf. [12] or [7]). Moreover, an R -module M is said to be *minimax*, if there exists a finitely generated submodule N of M , such that M/N is Artinian. The class of minimax modules was introduced by H. Zöschinger [14] and he has given in [14, 15] many equivalent conditions for a module to be minimax.

2. MAIN RESULTS

In [10], P. H. Quy introduced the class of FSF modules and he has given some properties of this modules. The R -module M is said to be a FSF module if there is a finitely generated submodule N of M such that the support of the quotient module M/N is finite. When R is a Noetherian ring, it is clear that, if M is FSF, then $\dim \operatorname{Supp} M/N \leq 1$. This motivates the following definition.

Definition 2.1. Let n be a non-negative integer. An R -module M is said to be in dimension $< n$, if there is a finitely generated submodule N of M such that $\dim \operatorname{Supp} M/N < n$.

Remark 2.2. Let n be a non-negative integer and let M be an R -module.

- (1) if $n = 0$, then M is in dimension $< n$ if and only if M is Noetherian.
- (2) If M is minimax, then M is in dimension < 1 . In particular, if M is Noetherian or Artinian, then M is in dimension < 1 .
- (3) If M is FSF, then M is in dimension < 2 .
- (4) If M is skinny, then M is in dimension < 2 , by [3, Theorem 3.3].
- (5) If M is reflexive, then M is in dimension < 1 .
- (6) If M is linearly compact, then M is in dimension < 1 .

Definition 2.3. If T is an arbitrary subset of $\operatorname{Spec} R$ and $n \in \mathbb{N}_0$, then we set

$$(T)_{\geq n} := \{\mathfrak{p} \in T \mid \dim R/\mathfrak{p} \geq n\}.$$

Definition 2.4. Let R be a Noetherian ring, \mathfrak{a} an ideal of R and M an R -module. For any non-negative integer n , we define

$$h_{\mathfrak{a}}^n(M) := \inf\{0 \leq i \in \mathbb{Z} : H_{\mathfrak{a}}^i(M) \text{ is not in dimension } < n\}.$$

Lemma 2.5. Let R be a Noetherian ring, n a non-negative integer and L an R -module such that $\dim L \geq n$. If there is a submodule L' of L such that $\dim L/L' \geq n$, then

$$\bigcap_{\mathfrak{p} \in (\operatorname{Ass}_R L)_{\geq n}} \mathfrak{p} \subseteq \bigcap_{\mathfrak{p} \in (\operatorname{Ass}_R L/L')_{\geq n}} \mathfrak{p}.$$

Proof. Let $\mathfrak{p} \in (\operatorname{Ass}_R L/L')_{\geq n}$. Then $\mathfrak{p} \in \operatorname{Supp} L$, and so there exists $\mathfrak{q} \in \operatorname{Ass}_R L$ such that $\mathfrak{q} \subseteq \mathfrak{p}$. Therefore $\mathfrak{q} \in (\operatorname{Ass}_R L)_{\geq n}$, and thus

$$\bigcap_{\mathfrak{p} \in (\operatorname{Ass}_R L)_{\geq n}} \mathfrak{p} \subseteq \mathfrak{q} \subseteq \mathfrak{p}.$$

This completes the proof. □

Lemma 2.6. Let R be a Noetherian ring, n a non-negative integer and L an R -module in dimension $< n$. Then the set $(\operatorname{Ass}_R L)_{\geq n}$ is finite.

Proof. Since L is in dimension $< n$, it follows from the definition that there is a finitely generated submodule L' of L such that $\dim \operatorname{Supp} L/L' < n$. Now, from the exact sequence

$$0 \longrightarrow L' \longrightarrow L \longrightarrow L/L' \longrightarrow 0$$

we obtain

$$(\operatorname{Ass}_R L)_{\geq n} \subseteq (\operatorname{Ass}_R L')_{\geq n} \cup (\operatorname{Ass}_R L/L')_{\geq n}.$$

As $\dim \text{Supp } L/L' < n$, it follows that $(\text{Ass}_R L/L')_{\geq n} = \emptyset$. Thus

$$(\text{Ass}_R L)_{\geq n} \subseteq (\text{Ass}_R L')_{\geq n},$$

and so the set $(\text{Ass}_R L)_{\geq n}$ is finite. \square

Lemma 2.7. *Let R be a Noetherian ring, s a non-negative integer, M a finitely generated R -module and \mathfrak{a} an ideal of R . Let $\mathbf{x} = x_1, \dots, x_s \in \mathfrak{a}$ be an M -regular sequence. Then*

$$f_{\mathfrak{a}}(M) \leq s + f_{\mathfrak{a}}(M/\mathbf{x}M).$$

Proof. The assertion follows easily by induction on s . \square

Corollary 2.8. *Let R be a Noetherian ring, s non-negative integer, M a finitely generated R -module and \mathfrak{a} an ideal of R . Let $\mathbf{x} = x_1, \dots, x_s \in \mathfrak{a}$ be an M -regular sequence. Then for any $n \in \mathbb{N}_0$,*

$$f_{\mathfrak{a}}^n(M) \leq s + f_{\mathfrak{a}}^n(M/\mathbf{x}M).$$

Proof. The assertion follows from the definition of $f_{\mathfrak{a}}^n(M)$ and induction on s . \square

The following proposition which plays a key role in this paper will serve to shorten the proof of the main theorems.

Proposition 2.9. *Let R be a Noetherian ring, n a non-negative integer, M a finitely generated R -module and \mathfrak{a} an ideal of R . Then for all $i < f_{\mathfrak{a}}^n(M)$, the R -module $H_{\mathfrak{a}}^i(M)$ is in dimension $< n$.*

Proof. We show that $H_{\mathfrak{a}}^i(M)$ is in dimension $< n$ by induction on i . The case $i = 0$ is clear, because $H_{\mathfrak{a}}^0(M)$ is finitely generated. So suppose that $i > 0$ and that the result has been proved for smaller values of i . By this inductive assumption, $H_{\mathfrak{a}}^j(M)$ is in dimension $< n$ for $j = 0, 1, \dots, i-1$, and it only remains for us to prove that $H_{\mathfrak{a}}^i(M)$ is in dimension $< n$. To this end, it follows from [6, Corollary 2.1.7 and Lemma 2.1.2] that $H_{\mathfrak{a}}^i(M) \cong H_{\mathfrak{a}}^i(M/H_{\mathfrak{a}}^0(M))$ and $M/H_{\mathfrak{a}}^0(M)$ is an \mathfrak{a} -torsion-free R -module. Hence we can (and do) assume that M is an \mathfrak{a} -torsion-free R -module.

We now use [6, Lemma 2.1.1] to deduce that \mathfrak{a} contains an element x which is M -regular. Let $t \in \mathbb{N}$. Then the exact sequence

$$0 \longrightarrow M \xrightarrow{x^t} M \longrightarrow M/x^t M \longrightarrow 0$$

induces a long exact sequence

$$\dots \longrightarrow H_{\mathfrak{a}}^i(M) \xrightarrow{x^t} H_{\mathfrak{a}}^i(M) \longrightarrow H_{\mathfrak{a}}^i(M/x^t M) \longrightarrow H_{\mathfrak{a}}^{i+1}(M) \xrightarrow{x^t} H_{\mathfrak{a}}^{i+1}(M) \longrightarrow \dots,$$

and so we obtain the exact sequence

$$0 \longrightarrow H_{\mathfrak{a}}^{i-1}(M)/x^t H_{\mathfrak{a}}^{i-1}(M) \longrightarrow H_{\mathfrak{a}}^{i-1}(M/x^t M) \longrightarrow (0 :_{H_{\mathfrak{a}}^i(M)} x^t) \longrightarrow 0. \quad (\dagger)$$

Since by Corollary 2.8,

$$f_{\mathfrak{a}}^n(M) \leq 1 + f_{\mathfrak{a}}^n(M/x^t M),$$

it follows from the inductive hypothesis that the R -module $H_a^j(M/x^t M)$ is in dimension $< n$, for all $0 \leq j \leq i-1$.

Now, in order to show that $H_a^i(M)$ is in dimension $< n$, suppose the contrary is true. Then, as $H_a^{i-1}(M/x^t M)$ is in dimension $< n$, it follows from the exact sequence (†) that the R -module $(0 :_{H_a^i(M)} x^t)$ is in dimension $< n$. Therefore, in view of Lemma 2.6 the set $(\text{Ass}_R(0 :_{H_a^i(M)} x^t))_{\geq n}$ is finite. Consequently the set $(\text{Ass}_R H_a^i(M))_{\geq n}$ is also finite. Let

$$(\text{Ass}_R H_a^i(M))_{\geq n} = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}.$$

Then, in view of the definition, the $R_{\mathfrak{p}_j}$ -module $(H_a^i(M))_{\mathfrak{p}_j}$ is finitely generated for all $1 \leq j \leq r$. Thus for all $1 \leq j \leq r$, there exists a finitely generated submodule N_j of $H_a^i(M)$ such that $(H_a^i(M))_{\mathfrak{p}_j} = (N_j)_{\mathfrak{p}_j}$. Set $K_1 = N_1 + \dots + N_r$. Then K_1 is a finitely generated submodule of $H_a^i(M)$ and we have

$$(\text{Ass}_R H_a^i(M)/K_1)_{\geq n} \cap (\text{Ass}_R H_a^i(M))_{\geq n} = \emptyset.$$

Since K_1 is a finitely generated submodule of $H_a^i(M)$, it follows that there exists a non-negative integer l such that $K_1 \subseteq (0 :_{H_a^i(M)} x^l)$, and so

$$(K_1 :_{H_a^i(M)} x) \subseteq (0 :_{H_a^i(M)} x^{l+1}).$$

Now as the R -module $(0 :_{H_a^i(M)} x^{l+1})$ is in dimension $< n$, it follows that the R -module $(K_1 :_{H_a^i(M)} x)/K_1$ is in dimension $< n$. Therefore, using again the above method, we see that the set $(\text{Ass}_R(H_a^i(M)/K_1))_{\geq n}$ is finite. Since

$$(\text{Ass}_R H_a^i(M)/K_1)_{\geq n} \cap (\text{Ass}_R H_a^i(M))_{\geq n} = \emptyset,$$

it follows from Lemma 2.5 that

$$\bigcap_{\mathfrak{p} \in (\text{Ass}_R H_a^i(M))_{\geq n}} \mathfrak{p} \subsetneq \bigcap_{\mathfrak{p} \in (\text{Ass}_R H_a^i(M)/K_1)_{\geq n}} \mathfrak{p}.$$

By using the method used in the above, there is a finitely generated submodule K_2/K_1 of $H_a^i(M)/K_1$ such that

$$(\text{Ass}_R H_a^i(M)/K_2)_{\geq n} \cap (\text{Ass}_R H_a^i(M)/K_1)_{\geq n} = \emptyset,$$

and so

$$\bigcap_{\mathfrak{p} \in (\text{Ass}_R H_a^i(M)/K_1)_{\geq n}} \mathfrak{p} \subsetneq \bigcap_{\mathfrak{p} \in (\text{Ass}_R H_a^i(M)/K_2)_{\geq n}} \mathfrak{p}.$$

Proceeding in the same way we can find a chain of ideals of R ,

$$\bigcap_{\mathfrak{p} \in (\text{Ass}_R H_a^i(M))_{\geq n}} \mathfrak{p} \subsetneq \bigcap_{\mathfrak{p} \in (\text{Ass}_R H_a^i(M)/K_1)_{\geq n}} \mathfrak{p} \subsetneq \bigcap_{\mathfrak{p} \in (\text{Ass}_R H_a^i(M)/K_2)_{\geq n}} \mathfrak{p} \subsetneq \dots,$$

which is not stable. Consequently, $H_a^i(M)$ is in dimension $< n$, as required. \square

Now we are prepared to state and prove the first main result of this paper, which shows that the least integer i such that $H_a^i(M)$ is not in dimension $< n$, equals to $\inf\{f_{aR_{\mathfrak{p}}}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \text{Supp } M/aM \text{ and } \dim R/\mathfrak{p} \geq n\}$. This generalizes the main result of Asadollahi-Naghipour [1].

Theorem 2.10. *Let R be a Noetherian ring, \mathfrak{a} an ideal of R and M a finitely generated R -module. Then for any non-negative integer n ,*

$$f_{\mathfrak{a}}^n(M) = h_{\mathfrak{a}}^n(M).$$

Proof. The assertion follows from the definition and Proposition 2.9. \square

As a first application of Theorem 2.10, we show that the first non-minimax (resp. non-skinny) local cohomology modules $H_{\mathfrak{a}}^i(M)$ of a finitely module M over a Noetherian ring R with respect to an ideal \mathfrak{a} is $h_{\mathfrak{a}}^1(M)$ (resp. $h_{\mathfrak{a}}^2(M)$).

Corollary 2.11. *Let R be a Noetherian ring, \mathfrak{a} an ideal of R and M a finitely generated R -module. Then*

- (i) $h_{\mathfrak{a}}^1(M) = \inf\{i \in \mathbb{N}_0 \mid H_{\mathfrak{a}}^i(M) \text{ is not minimax}\}.$
- (ii) $h_{\mathfrak{a}}^2(M) = \inf\{i \in \mathbb{N}_0 \mid H_{\mathfrak{a}}^i(M) \text{ is not skinny}\},$ *whenever R is semilocal.*

Proof. (i) follows from [4, Corollary 2.4] and Theorem 2.10. To prove (ii) use [4, Proposition 3.7] and Theorem 2.10. \square

Proposition 2.12. *Let R be a Noetherian ring and n a non-negative integer. Let*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of R -modules. Then M is in dimension $< n$ if and only if M' and M'' are both in dimension $< n$.

Proof. We may suppose for the proof that M' is a submodule of M and that $M'' = M/M'$. If M is in dimension $< n$, then it is easy to verify that M' and M/M' are in dimension $< n$. Now, suppose that M' and M/M' are in dimension $< n$. Then there exists a finitely generated submodule T of M' such that $\dim \text{Supp } M'/T < n$. Let $N' = M'/T$ and $N = M/T$. Then we obtain the exact sequence

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N/N' \longrightarrow 0,$$

where $\dim \text{Supp } N' < n$ and N/N' is in dimension $< n$, (note that $N/N' \cong M/M'$). Now, since N/N' is in dimension $< n$ it follows from the definition that there is a finitely generated submodule L/N' of N/N' such that $\dim \text{Supp } N/L < n$. As L/N' is finitely generated, it follows that $L = N' + K$ for some finitely generated submodule K of L . Then it follows from $L/K \cong N'/K \cap N'$ that $\dim \text{Supp } L/K < n$. Therefore the exact sequence

$$0 \longrightarrow L/K \longrightarrow N/K \longrightarrow N/L \longrightarrow 0$$

implies that $\dim \text{Supp } N/K < n$. Consequently N is in dimension $< n$. Since $N = M/T$, it follows that $K = S/T$ for some submodule S of M containing T . As T and K are finitely generated, we deduce that S is also finitely generated. Now Because

$$M/S \cong (M/T)/(S/T) = N/K,$$

it yields that $\dim \operatorname{Supp} M/S < n$, and so by definition, M is in dimension $< n$ and the claim is true. \square

Before bringing the next results, let us recall that a full subcategory \mathcal{S} of the category of R -modules is called a *Serre subcategory*, when it is closed under taking submodules, quotients and extensions. One can easily check that the subcategories of, finitely generated, minimax, skinny, and Matlis reflexive modules are examples of Serre subcategory. The following result provides a new class of Serre subcategory.

Corollary 2.13. *For any non-negative integer n , the class of in dimension $< n$ modules over a Noetherian ring R consists a Serre subcategory of the category of R -modules.*

Proof. The assertion follows immediately from Proposition 2.12. \square

Corollary 2.14. *Let R be a Noetherian ring and n a non-negative integer. Then any quotient of an in dimension $< n$ module, as well as any finite direct sum of in dimension $< n$ modules, is in dimension $< n$.*

Proof. The assertion follows from definition and Proposition 2.12. \square

Corollary 2.15. *Let R be a Noetherian ring, \mathfrak{a} an ideal of R and M a finitely generated R -module. For a non-negative integer n , let $t = f_{\mathfrak{a}}^n(M)$. Then for each submodule N of $\oplus_{i=0}^{t-1} H_{\mathfrak{a}}^i(M)$, the set*

$$(\operatorname{Ass}_R(\oplus_{i=0}^{t-1} H_{\mathfrak{a}}^i(M)/N))_{\geq n}$$

is finite.

Proof. Since by Proposition 2.9, $\oplus_{i=0}^{t-1} H_{\mathfrak{a}}^i(M)/N$ is in dimension $< n$, the assertion follows from Lemma 2.6. \square

Corollary 2.16. *Let R be a Noetherian ring and n a non-negative integer. Let M, N be R -modules such that M is finitely generated and N is in dimension $< n$. Then $\operatorname{Ext}_R^i(M, N)$ and $\operatorname{Tor}_i^R(M, N)$ are in dimension $< n$ modules for all i . In particular, for any ideal \mathfrak{a} of R , the R -modules $\operatorname{Ext}_R^i(R/\mathfrak{a}, N)$ and $\operatorname{Tor}_i^R(R/\mathfrak{a}, N)$ are in dimension $< n$, for all i .*

Proof. As R is Noetherian and M is finitely generated, it follows that M possesses a free resolution

$$\mathbb{F}_{\bullet} : \cdots \rightarrow F_s \rightarrow F_{s-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow 0,$$

whose free modules have finite ranks.

Thus $\operatorname{Ext}_R^i(M, N) = H^i(\operatorname{Hom}_R(\mathbb{F}_{\bullet}, N))$ is a subquotient of a direct sum of finitely many copies of N . Therefore, it follows from Corollary 2.14 that $\operatorname{Ext}_R^i(M, N)$ is in dimension $< n$ for all $i \geq 0$. By using a similar proof as above we can deduce that $\operatorname{Tor}_i^R(M, N)$ is in dimension $< n$ for all $i \geq 0$. \square

The following proposition is needed in the proof of the second main theorem of this section.

Proposition 2.17. *Let R be a Noetherian ring and n a non-negative integer. Let M be a finitely generated R -module and N an arbitrary R -module. Let t be a non-negative integer such that $\text{Ext}_R^i(M, N)$ is in dimension $< n$ for all $i \leq t$. Then for any finitely generated R -module L with $\text{Supp } L \subseteq \text{Supp } M$, $\text{Ext}_R^i(L, N)$ is in dimension $< n$ for all $i \leq t$.*

Proof. Since $\text{Supp } L \subseteq \text{Supp } M$, it follows from the Gruson's Theorem (cf. [13, Theorem 4.1]), that there exists a chain

$$0 = L_0 \subset L_1 \subset \cdots \subset L_k = L,$$

such that the factors L_j/L_{j-1} are homomorphic images of a direct sum of finitely many copies of M . Now consider the exact sequences

$$0 \rightarrow K \rightarrow M^r \rightarrow L_1 \rightarrow 0$$

$$0 \rightarrow L_1 \rightarrow L_2 \rightarrow L_2/L_1 \rightarrow 0$$

$$\vdots$$

$$0 \rightarrow L_{k-1} \rightarrow L_k \rightarrow L_k/L_{k-1} \rightarrow 0,$$

for some positive integer r .

Now from the long exact sequence

$$\cdots \rightarrow \text{Ext}_R^{i-1}(L_{j-1}, N) \rightarrow \text{Ext}_R^i(L_j/L_{j-1}, N) \rightarrow \text{Ext}_R^i(L_j, N) \rightarrow \text{Ext}_R^i(L_{j-1}, N) \rightarrow \cdots,$$

and an easy induction on k , it suffices to prove the case when $k = 1$.

Thus there is an exact sequence

$$0 \rightarrow K \rightarrow M^r \rightarrow L \rightarrow 0 \quad (*)$$

for some $r \in \mathbb{N}$ and some finitely generated R -module K .

Now, we use induction on t . First, $\text{Hom}_R(L, N)$ is a submodule of $\text{Hom}_R(M^r, N)$; hence in view of assumption and Corollary 2.14, $\text{Ext}_R^0(L, N)$ is in dimension $< n$. So assume that $t > 0$ and that $\text{Ext}_R^j(L', N)$ is in dimension $< n$ for every finitely generated R -module L' with $\text{Supp } L' \subseteq \text{Supp } M$ and all $j \leq t - 1$. Now, the exact sequence $(*)$ induces the long exact sequence

$$\cdots \rightarrow \text{Ext}_R^{i-1}(K, N) \rightarrow \text{Ext}_R^i(L, N) \rightarrow \text{Ext}_R^i(M^r, N) \rightarrow \cdots,$$

so that, by the inductive hypothesis, $\text{Ext}_R^{i-1}(K, N)$ is in dimension $< n$ for all $i \leq t$. On the other hand, according to Corollary 2.14,

$$\text{Ext}_R^i(M^r, N) \cong \bigoplus^r \text{Ext}_R^i(M, N)$$

is in dimension $< n$. Therefore, it follows from Proposition 2.12 that $\text{Ext}_R^i(L, N)$ is in dimension $< n$ for all $i \leq t$, and this completes the inductive step. \square

Corollary 2.18. *Let R be a Noetherian ring and \mathfrak{a} an ideal of R . Assume that t, n are non-negative integers. Then, for any R -module M the following conditions are equivalent:*

- (i) $\text{Ext}_R^i(R/\mathfrak{a}, M)$ is in dimension $< n$ for all $i \leq t$.
- (ii) For any ideal \mathfrak{b} of R with $\mathfrak{b} \supseteq \mathfrak{a}$, $\text{Ext}_R^i(R/\mathfrak{b}, M)$ is in dimension $< n$ for all $i \leq t$.
- (iii) For any finitely generated R -module N with $\text{Supp}(N) \subseteq V(\mathfrak{a})$, $\text{Ext}_R^i(N, M)$ is in dimension $< n$ for all $i \leq t$.
- (iv) For any minimal prime ideal \mathfrak{p} over \mathfrak{a} , $\text{Ext}_R^i(R/\mathfrak{p}, M)$ is in dimension $< n$ for all $i \leq t$.

Proof. In view of the Proposition 2.17, it is enough to show that (iv) implies (i). To do this, let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ be the minimal primes of \mathfrak{a} . Then, by assumption the R -modules $\text{Ext}_R^i(R/\mathfrak{p}_j, M)$ are in dimension $< n$ for all $j = 1, 2, \dots, s$. Hence by Corollary 2.14,

$$\bigoplus_{j=1}^s \text{Ext}_R^i(R/\mathfrak{p}_j, M) \cong \text{Ext}_R^i(\bigoplus_{j=1}^s R/\mathfrak{p}_j, M)$$

is in dimension $< n$. Since $\text{Supp}(\bigoplus_{j=1}^s R/\mathfrak{p}_j) = \text{Supp} R/\mathfrak{a}$, it follows from Proposition 2.17 that $\text{Ext}_R^i(R/\mathfrak{a}, M)$ is in dimension $< n$, as required. \square

Following we let \mathcal{S} denote a Serre subcategory of the category of R -modules.

Lemma 2.19. *Let R be a Noetherian ring and \mathfrak{a} an ideal of R . Let s be a non-negative integer and M an R -module such that $\text{Ext}_R^j(R/\mathfrak{a}, M) \in \mathcal{S}$. If $\text{Ext}_R^j(R/\mathfrak{a}, H_{\mathfrak{a}}^i(M)) \in \mathcal{S}$ for all $i < s$ and all $j \geq 0$, then $\text{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^s(M)) \in \mathcal{S}$.*

Proof. See [2, Theorem 2.2]. \square

Proposition 2.20. *Let R be a Noetherian ring and \mathfrak{a} an ideal of R . Let s be a non-negative integer and let M be an R -module such that $\text{Ext}_R^{s+1}(R/\mathfrak{a}, M) \in \mathcal{S}$. If $\text{Ext}_R^j(R/\mathfrak{a}, H_{\mathfrak{a}}^i(M)) \in \mathcal{S}$ for all $i < s$ and all $j \geq 0$, then $\text{Ext}_R^1(R/\mathfrak{a}, H_{\mathfrak{a}}^s(M)) \in \mathcal{S}$.*

Proof. We use induction on s . Let $s = 0$. Then the exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(M) \longrightarrow M \longrightarrow M/\Gamma_{\mathfrak{a}}(M) \longrightarrow 0,$$

induces the exact sequence

$$\text{Hom}_R(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M)) \longrightarrow \text{Ext}_R^1(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M)) \longrightarrow \text{Ext}_R^1(R/\mathfrak{a}, M).$$

As $\text{Hom}_R(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M)) = 0$ and $\text{Ext}_R^1(R/\mathfrak{a}, M) \in \mathcal{S}$, it follows that $\text{Ext}_R^1(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M))$ is also in \mathcal{S} .

Now suppose inductively that $s > 0$ and that the assertion holds for $s - 1$. Again using the exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(M) \longrightarrow M \longrightarrow M/\Gamma_{\mathfrak{a}}(M) \longrightarrow 0,$$

for all $j \geq 0$, we obtain the following exact sequence,

$$\text{Ext}_R^j(R/\mathfrak{a}, M) \longrightarrow \text{Ext}_R^j(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M)) \longrightarrow \text{Ext}_R^{j+1}(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M)).$$

Now since by assumption, $\text{Ext}_R^{s+2}(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M))$ and $\text{Ext}_R^{s+1}(R/\mathfrak{a}, M)$ are in \mathcal{S} , it follows that $\text{Ext}_R^{s+1}(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M)) \in \mathcal{S}$. Also, it follows easily from assumption and [6, Corollary 2.1.7] that

$$\text{Ext}_R^j(R/\mathfrak{a}, H_{\mathfrak{a}}^i(M/\Gamma_{\mathfrak{a}}(M))) \in \mathcal{S}$$

for all $i < s$ and all $j \geq 0$. Therefore we may assume that $\Gamma_{\mathfrak{a}}(M) = 0$.

Next, let $E_R(M)$ denote the injective hull of M . Then $\Gamma_{\mathfrak{a}}(E_R(M)) = 0$, and so it follows from the exact sequence

$$0 \longrightarrow M \longrightarrow E_R(M) \longrightarrow E_R(M)/M \longrightarrow 0,$$

that $H_{\mathfrak{a}}^{i+1}(M) \cong H_{\mathfrak{a}}^i(E_R(M)/M)$ for all $i \geq 0$. Also, as $\text{Hom}_R(R/\mathfrak{a}, E_R(M)) = 0$, it yields that

$$\text{Ext}_R^j(R/\mathfrak{a}, E_R(M)/M) \cong \text{Ext}_R^{j+1}(R/\mathfrak{a}, M),$$

for all $j \geq 0$. Consequently the R -module $E_R(M)/M$ satisfies our condition hypothesis. Thus

$$\text{Ext}_R^1(R/\mathfrak{a}, H_{\mathfrak{a}}^{s-1}(E_R(M)/M)) \in \mathcal{S},$$

and so the assertion follows from $H_{\mathfrak{a}}^s(M) \cong H_{\mathfrak{a}}^{s-1}(E_R(M)/M)$. \square

We are now ready to state and prove the second main result of this paper which is a generalization of the main result of Quy [10, Theorem 3.2] and Brodmann-Lashgari [5, Theorem 2.2].

Theorem 2.21. *Let R be a Noetherian ring, \mathfrak{a} an ideal of R and M a finitely generated R -module. For a non-negative integer n , let $t = f_{\mathfrak{a}}^n(M)$. Then the following statements hold:*

- (i) *The R -modules $\text{Ext}_R^i(R/\mathfrak{a}, H_{\mathfrak{a}}^i(M))$ are in dimension $< n$ for $i = 0, 1, \dots, t-1$ and all integers j .*
- (ii) *The R -modules $\text{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^t(M))$ and $\text{Ext}_R^1(R/\mathfrak{a}, H_{\mathfrak{a}}^t(M))$ are in dimension $< n$.*
- (iii) *For each finitely generated R -module N with $\text{Supp}(N) \subseteq V(\mathfrak{a})$ the R -modules $\text{Ext}_R^j(N, H_{\mathfrak{a}}^i(M))$ are in dimension $< n$ for all $i = 0, 1, \dots, t-1$ and for all integers j .*
- (iv) *The set $(\text{Ass}_R(H_{\mathfrak{a}}^t(M)))_{\geq n}$ is finite.*
- (v) *The set $\text{Ass}_R(\oplus_{i=0}^t H_{\mathfrak{a}}^i(M)/K)_{\geq n}$ is finite, for any finitely generated submodule K of $\oplus_{i=0}^t H_{\mathfrak{a}}^i(M)$.*

Proof. (i) This follows immediately from Proposition 2.9 and Corollary 2.16. In order to show (ii) use part (i), Lemma 2.19, Theorem 2.10 and Proposition 2.20. Part (iii) follows from Proposition 2.17 and part (i). Moreover, (iv) follows readily from Lemma 2.6, part (ii) and the fact that

$$\text{Ass}_R \text{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^t(M)) = \text{Ass}_R H_{\mathfrak{a}}^t(M).$$

Finally, (v) follows easily from Corollary 2.15, part (iv) and the exact sequence

$$\text{Hom}_R(R/\mathfrak{a}, \oplus_{i=0}^t H_{\mathfrak{a}}^i(M)) \longrightarrow \text{Hom}_R(R/\mathfrak{a}, \oplus_{i=0}^t H_{\mathfrak{a}}^i(M)/K) \longrightarrow \text{Ext}_R^1(R/\mathfrak{a}, K).$$

\square

The paper ends a result about the finiteness Bass numbers of a certain local cohomology modules. Recall that for any prime ideal \mathfrak{p} of R and an R -module L , the i -th Bass number $\mu^i(\mathfrak{p}, L)$ is defined to be $\dim_{k(\mathfrak{p})} \text{Ext}_{R_{\mathfrak{p}}}^i(k(\mathfrak{p}), L_{\mathfrak{p}})$, where $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$.

Corollary 2.22. *Let R be a Noetherian ring, \mathfrak{a} an ideal of R and M a finitely generated R -module. For a non-negative integer n , let $t = h_{\mathfrak{a}}^n(M)$. Let $\mathfrak{p} \in \text{Spec } R$ be such that $\dim R/\mathfrak{p} \geq n$. Then*

- (i) *The Bass numbers $\mu^j(\mathfrak{p}, H_{\mathfrak{a}}^i(M))$ are finite for all $0 \leq i \leq t - 1$ and all integers j .*
- (ii) *The Bass numbers $\mu^j(\mathfrak{p}, H_{\mathfrak{a}}^t(M))$ are finite for $j = 0, 1$.*

Proof. The part (i) follows from Theorem 2.10 and the fact that the Bass numbers of finitely generated modules are finite. Also, (ii) follows from Theorem 2.10 and [11, Corollary 3.5]. \square

Acknowledgments

The authors are deeply grateful to the referee for his or her valuable suggestions on the paper and for drawing the authors' attention to Corollary 2.11. Also, we would like to thank Dr. Kamal Bahmanpour for reading of the first draft and valuable discussions. Finally, the authors would like to thank Tabriz branch, Islamic Azad University for the financial support of this research, which is based on a research project contract

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DEPARTMENT OF MATHEMATICS, TABRIZ BRANCH, ISLAMIC AZAD UNIVERSITY, TABRIZ, IRAN
E-mail address: amehrvarz2013@gmail.com

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TABRIZ, TABRIZ, IRAN.
E-mail address: naghipour@ipm.ir
E-mail address: naghipour@tabrizu.ac.ir

DEPARTMENT OF MATHEMATICS, AZARBAIJAN SHAHID MADANI UNIVERSITY, TABRIZ, IRAN.
E-mail address: sedghi@azaruniv.ac.ir
E-mail address: m_sedghi@tabrizu.ac.ir